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ON THE MOST GENERAL PLANE CLOSED POINT-SET THROUGH WHICH IT IS POSSIBLE TO PASS A SIMPLE CONTINUOUS ARC.*

By R. L. Moore and J. R. Kline.

A set of points is said to be totally disconnected if it contains no connected subset consisting of more than one point. In 1905 L. Zoretti† showed that every closed, bounded and totally disconnected set of points is a subset of a Cantorean‡ line. In 1906 F. Riesz§ attempted to show that every such set of points is a subset of a simple continuous arc.|| Shortly thereafter Zoretti¶ pointed out that Riesz's argument was fallacious. He, however, left unsettled the question whether Riesz's theorem was true or false. In 1910, in an article that contains no reference either to Riesz or to Zoretti, Denjoy** indicated that this theorem could be proved with the use of certain ideas contained in a former paper†† of his own. We have not, however, succeeded in determining from his meager indications just what sort of argument he had in mind. At any rate, in order that a closed and bounded point-set should be a subset of a simple continuous arc it is of course not *necessary* that it should be totally disconnected. In the present paper we will establish the following result.

Theorem 1. *In order that a closed and bounded point-set M should be a subset of a simple continuous arc it is necessary and sufficient that every closed, connected subset of M should be either a single point or a simple continuous arc t such that no point of t, with the exception of its endpoints is a limit point of $M - t$.*‡‡

* Presented to the American Mathematical Society, February 24, 1917.

† Sur les fonctions analytiques uniformes, Journal de Mathematiques pures et appliquées, vol. 1 (1905), p. 12.

‡ A Cantorean line is a closed connected point-set that contains the interior of no circle.

§ Sur les ensembles discontinus, Comptes Rendus, vol. 141 (1905), pp. 650–655.

|| It is well known that not every Cantorean line is a simple continuous arc.

¶ Sur les ensembles discontinus, Comptes Rendus, vol. 142 (1906), pp. 763–764. Riesz made use of the false proposition that the orthogonal projection of a closed totally disconnected point-set is itself necessarily disconnected.

** Continu et discontinu, Comptes Rendus, vol. 151 (1910), pp. 138–140.

†† Sur les ensembles parfaits discontinus, Comptes Rendus, vol. 149 (1909), pp. 1048–1050.

‡‡ Consider the point-set $\overline{M}$ (Fig. 1) composed of the straight interval t from (0, 0) to (2, 0) together with the infinite set of points (1, 1), (1, 1/2), (1, 1/3), (1, 1/4), $\cdots$. Every closed connected subset of $\overline{M}$ is either a single point or a simple continuous arc. But the point (1, 0) of

In our proof of this theorem we will make use of the following lemmas.

LEMMA 1. *If G is a finite set of simple closed curves and M is a closed point-set and each point of M is within some curve of the set G_1 but no point of M is on any curve of the set G, then there exists a finite set $\bar{G}$ of simple closed curves such that* 1) *each point of M is within some curve of $\bar{G}$,* 2) *every curve of the set $\bar{G}$ lies entirely without every other curve of the set $\bar{G}$,* 3) *each curve of the set of $\bar{G}$ is within some curve of the set G_1.*

Proof. If P is a point of M there exists* a closed curve $\bar{J}_P$ such that (1) every point of $\bar{J}_P$ is on some curve of the set G, (2) every point within $\bar{J}_P$ is within every curve of the set G that encloses P and without every curve of G that does not enclose P. The set M_P of all those points of M that lie within $\bar{J}_P$ is closed. It follows† that there exists a closed curve J_P that lies within $\bar{J}_P$ and encloses M_P. Such a curve will be said to *properly* enclose P. It is clear that if J_{P_1} and J_{P_2} are curves that properly enclose the points P_1 and P_2 respectively, then J_{P_1} either lies entirely without J_{P_2} or properly encloses every point of M that is enclosed by J_{P_2}. For each point P of M select just one J_P and let K denote the set of curves thus obtained. By the Heine-Borel Theorem there is a finite subset $\bar{G}$ of the set of curves K such that every point of M is within some curve of the set $\bar{G}$. The set $\bar{G}$ satisfies the conditions of Lemma 1.

LEMMA 2. *Suppose that M is a closed and bounded set of points and G is a set of closed curves such that* (1) *every point of M is either on or within*

the arc t, though not an endpoint of t, is a limit point of the set of points $M - t$. It is accordingly impossible to pass a simple continuous arc through $\bar{M}$.

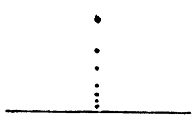

FIG. 1.

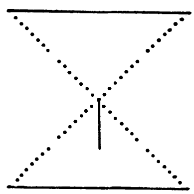

FIG. 2.

Consider, on the other hand, the set of points M (Fig. 2) composed of straight intervals from $(-1, 1)$ to $(1, 1)$, from $(-1, -1)$ to $(-1, 1)$ and from $(0, 0)$ to $(0, -1/2)$ respectively, together with the eight infinite sets $(1/2, 1/2)$, $(1/3, 1/3)$, $(1/4, 1/4)$, $\cdots$, $(1/2, 1/2)$, $(2/3, 2/3)$, $(3/4, 3/4)$, $\cdots$, $(-1/2, 1/2)$, $(-1/3, 1/3)$, $(-1/4, 1/4)$, $\cdots$, $(-1/2, 1/2)$, $(-2/3, 2/3)$, $(-3/4, 3/4)$, $\cdots$, $(-1/2, -1\ 2)$, $(-1/3, -1/3)$, $(-1/4, -1/4)$, $\cdots$, $(-1/2, -1/2)$, $(-2/3, -2/3)$, $(-3/4, -3/4)$, $\cdots$, $(1/2, -1/2)$, $(1/3, -1/3)$, $(1/4, -1/4)$, $\cdots$ and $(1/2, -1/2)$, $(2/3, -2/3)$, $(3/4, -3/4)$, $\cdots$. This set of points M satisfies the conditions of Theorem 1. It is accordingly a subset of a simple continuous arc.

* The existence of $\bar{J}_P$ can be proved with the use of Theorems 37 and 38 of R. L. Moore's paper On the foundations of plane analysis situs, Transactions of the American Mathematical Society, vol. 17 (1916), pp. 131–164. Hereafter this paper will be referred to as "Foundations."

† Cf. Theorem 46 of Foundations.

some curve of G, (2) if a point of M is not within any curve of G then there exists a curve g of the set G such that P is on g and such that, if P is a limit point of any subset of M, then every such subset of M contains points within g. Then there is a finite set of curves $\bar{G}$ such that every curve of $\bar{G}$ is a curve of G and such that $\bar{G}$ satisfies with respect to M the same conditions that are assumed above as being satisfied by G.

Proof. If P is a point of M that does not lie within any curve of the set G there exists a closed curve J_P belonging to G and containing P such that P is not a limit point of any point-set that contains no points within J_P. There exists a closed curve C_P enclosing P such that every point of M within C_P, except the point P, is within J_P. Let K denote the set of all such curves C_P for all such points P. Every point of M is within a curve of the set $G + K$. By the Heine Borel Theorem there exists a finite set of curves C_{P_1}, C_{P_2}, C_{P_3}, $\cdots$, C_{P_n}, belonging to K, and a finite set g_1, g_2, g_3, $\cdots$, g_m, belonging to G, such that every point of M is within a curve of one or the other of these two sets. It is clear that the set $\bar{G}$ of curves g_1, g_2, g_3, $\cdots$, g_m, J_{P_1}, J_{P_2}, J_{P_3}, $\cdots$, J_{P_n} satisfies the requirements of Lemma 2.

Proof of Theorem 1. In order that a closed and bounded set of points should be a subset of a simple continuous arc it is evidently *necessary* that it should satisfy the conditions imposed in the statement of Theorem 1. We will proceed to show that these conditions are also *sufficient*. Suppose that M is a point-set that fulfills all of these conditions. An arc will be called a *maximum arc* if it belongs to M but is not a subset of any other arc that belongs to M. If A and B are the endpoints of such a maximum arc AB, it can be proved with the use of methods employed in the proof of Theorem 32 of Foundations that there exist two arcs AXB and AYB, with no point in common except the points A and B, such that the closed curve $AXBYA$ formed by these arcs encloses every point of the arc AB except the points A and B but encloses no point of M that is not on the arc AB. By a theorem due to Zoretti* there exists a closed curve J_A that encloses A but not B, lies within a circle of radius 1 and does not contain B or any point of M that is not on the arc AB. By Theorem 43 of Foundations there exists a simple closed curve a containing A such that every point of a belongs either to $AXBYA$ or to J_A and such that every point within a is without $AXBYA$ and within J_A. The curve a contains the point A but no other point of M, encloses no point of AB and lies within a circle of radius 1. Similarly there exists a closed curve b that lies entirely without a, contains B but no other point of M and lies within a circle of radius 1. Each point P of M that is not a proper part of a connected

* Sur les fonctions analytiques uniformes, loc. cit., pp. 9–11.

subset of M can be enclosed by a simple closed curve p that lies within a circle of radius 1 and has no point in common with M. The point-set composed of all endpoints of maximum arcs, together with all those points of M that are not proper parts of connected subsets of M, is a closed set of points. It follows by Lemma 2 that there exists a finite set $\bar{G}$ of closed curves $\bar{a}_1, \bar{a}_2, \bar{a}_3, \cdots, \bar{a}_{\bar{n}}, \bar{b}_1, \bar{b}_2, \bar{b}_3, \cdots, \bar{b}_{\bar{n}}, \bar{p}_1, \bar{p}_2, \bar{p}_3, \cdots, \bar{p}_{\bar{m}}$ and a set $\bar{\tau}$ of maximum arcs $\bar{A}_1\bar{B}_1, \bar{A}_2\bar{B}_2, \cdots, \bar{A}_n\bar{B}_n$ such that every point of M is either within a curve of the set $\bar{G}$ or on an arc of the set $\bar{\tau}$ and such that for every i $(1 \leqq i \leqq \bar{n})$ and j $(1 \leqq j \leqq \bar{m})$ (1) each of the curves $\bar{a}_i$, $\bar{b}_i$, $\bar{p}_j$ lies within a circle of radius 1, (2) $\bar{a}_i$ contains $\bar{A}_i$ and $\bar{b}_i$ contains $\bar{B}_i$ but neither of them contains any other point of M, (3) $\bar{a}_i$ lies entirely without $\bar{b}_i$, (4) the arc $\bar{A}_i\bar{B}_i$ lies entirely without $\bar{a}_i$ and entirely without $\bar{b}_i$ except that $\bar{A}_i$ and $\bar{B}_i$ are on $\bar{a}_i$ and $\bar{b}_i$ respectively, (5) $\bar{A}_i\bar{B}_i$ is a maximum arc, (6) $\bar{p}_j$ contains no point of M.

For each i $(1 \leqq i \leqq \bar{n})$ there exists, with center at $\bar{A}_i$ and with radius less than 1, a circle $\bar{C}_i$ which neither encloses nor contains a point of any curve of the set $\bar{G}$ except the curve $\bar{a}_i$, and which does not enclose every point of $\bar{a}_i$. Let O_i denote a point of $\bar{a}_i$ that is not within $\bar{C}_i$ and let $\bar{E}_i$ and $\bar{F}_i$ denote points in the order $O_i\bar{E}_i\bar{A}_i\bar{F}_i$ on $\bar{a}_i$. By Zoretti's theorem* there exists within $\bar{C}_i$ a closed curve $\bar{J}_i$ that encloses $\bar{A}_i$ and contains no point of $M - \bar{A}_i\bar{B}_i$. There exists an interval $\bar{X}_i\bar{Y}_i\bar{Z}_i$ of $\bar{J}_i$ that lies entirely within $\bar{a}_i$ except that its endpoints $\bar{X}_i$ and $\bar{Z}_i$ are on the segments $O_i\bar{E}_i\bar{A}_i$ and $O_i\bar{F}_i\bar{A}_i$ respectively of $\bar{a}_i$. The arc $\bar{X}_i\bar{Y}_i\bar{Z}_i$ and the interval $\bar{X}_i\bar{A}_i\bar{Z}_i$ of the closed curve $\bar{a}_i$ form a closed curve a_i' that neither encloses nor contains any point of any curve of the set $\bar{G}$ except $\bar{a}_i$ and has on it no point of M except the point $\bar{A}_i$. There exists an arc $X_i'Y_i'Z_i'$ lying entirely within $\bar{a}_i$ except that its endpoints X_i' and Z_i' are on $\bar{a}_i$ in the order $X_i'\bar{X}_i\bar{A}_i\bar{Z}_iZ_i'$ and such that (1) $X_i'Y_i'Z_i'$ contains no point of M or of any curve of the set $\bar{G}$, (2) every point of M that is within $\bar{a}_i$ is either within a_i' or within the closed curve a_i'' formed by $X_i'Y_i'Z_i'$ and that interval of $\bar{a}_i$ whose endpoints are X_i' and Z_i' and which does not contain $\bar{A}_i$. Similarly, for each i there exist two closed curves b_i' and b_i'' such that (1) every point of M that lies within $\bar{b}_i$ is within either b_i' or b_i'', (2) the curve b_i' contains $\bar{B}_i$ but no other point of M, (3) the curve b_i'' contains no point of M, (4) no curve, except $\bar{b}_i$, of the set $\bar{G}$ contains any point on or within b_i', (5) the interiors of b_i' and b_i'' are subsets of the interior of $\bar{b}_i$. No one of the set $\bar{G}'$ of curves $a_1', a_2', \cdots, a_{\bar{n}}', b_1', b_2', \cdots, b_{\bar{n}}'$ contains or encloses any point of any other curve of the set $\bar{G}'$ or of the set G'' of curves $a_1'', a_2'', \cdots, a_{\bar{n}}'', b_1'', b_2'', \cdots, b_{\bar{n}}'', \bar{p}_1, \bar{p}_2, \cdots, \bar{p}_{\bar{m}}$. If a curve of G'' encloses a point of $\bar{a}_i$ or of $\bar{b}_i$ it must enclose $\bar{a}_i$, $\bar{b}_i$ and $\bar{A}_i\bar{B}_i$. The set

* Loc. cit., pp. 9–11.

$\bar{G}'$ has as a subset a set G' of closed curves a_{m_1}', a_{m_2}', $\cdots$, a_{m_n}', b_{m_1}', b_{m_2}', $\cdots$, b_{m_n}' such that every point of M is either within some curve of G' or G'' or on some arc of the set $\bar{A}_{m_1}\bar{B}_{m_1}$, $\bar{A}_{m_2}\bar{B}_{m_2}$, $\cdots$, $\bar{A}_{m_n}\bar{B}_{m_n}$ and such that no curve of the set G'' either contains or encloses a point of any curve of the set G'. No curve of the set G'' contains a point of M. It follows with the aid of Lemma 1 that there exists a finite set of closed curves p_1, p_2, $\cdots$, p_m such that (1) every point of M that is within a curve of the set G'' is also within a curve of the set p_1, p_2, $\cdots$, p_m, (2) every curve of the set p_1, p_2, $\cdots$, p_m is without every other curve of this set and within some curve of the set G'' and therefore without every curve of the set G'. Replace the symbols a_{m_i}' by a_i, b_{m_i}' by b_i, $\bar{A}_{m_i}$ by A_i and $\bar{B}_{m_i}$ by B_i. We now have a set G of closed curves a_1, a_2, $\cdots$, a_n, b_1, b_2, $\cdots$, b_n, p_1, p_2, $\cdots$, p_m and a set Q of arcs A_1B_1, A_2B_2, $\cdots$, A_nB_n such that (1) every point of M is either within a curve of the set G or on an arc of the set Q, (2) no two arcs of the set Q have a point in common, (3) for every i $(1 \leqq i \leqq n)$ the arc A_iB_i lies entirely without every curve of the set G except that A_i is on a_i and B_i is on b_i, (4) each curve of the set G is within a circle of radius 1, (5) every curve of the set G is entirely without every other curve of the set G. There exists a set $\bar{Q}$ of arcs C_1D_2, C_2D_3, C_3D_4, $\cdots$, $C_{n-1}D_n$, C_nE_1, F_1E_2, F_2E_3, F_3E_4, $\cdots$, $F_{m-1}E_m$ such that (1) $C_i (1 \leqq i \leqq n)$ is a point of b_i distinct from B_i, D_i $(2 \leqq i \leqq n)$ is a point of a_i distinct from A_i, E_i $(1 \leqq i \leqq m)$ is a point of p_i, F_i $(1 \leqq i \leqq m - 1)$ is a point of p_i distinct from E_i, (2) no arc of the set $\bar{Q}$ has a point in common with any other arc of the set $\bar{Q}$ or any arc of the set Q, (3) every arc of Q lies, except for its endpoints, entirely without every curve of the set G. The arcs of Q and of $\bar{Q}$ and the curves of G form an arc-curve chain K_1* covering M. For every $i (1 \leqq i \leqq n)$ there exists an arc-curve chain $\left\{\begin{smallmatrix}\beta_i\\ \alpha_i\end{smallmatrix}\right\}$ such that (1) $\left\{\begin{smallmatrix}\beta_i\\ \alpha_i\end{smallmatrix}\right\}$ covers the set of all those points of M that lie within $\left\{\begin{smallmatrix}b_i\\ a_i\end{smallmatrix}\right\}$, (2) the $\left\{\begin{smallmatrix}\text{first}\\ \text{last}\end{smallmatrix}\right\}$ curve of $\left\{\begin{smallmatrix}\beta_i\\ \alpha_i\end{smallmatrix}\right\}$ contains the point $\left\{\begin{smallmatrix}B_i\\ A_i\end{smallmatrix}\right\}$ but lies except for this point entirely within $\left\{\begin{smallmatrix}b_i\\ a_i\end{smallmatrix}\right\}$, (3) every arc of $\left\{\begin{smallmatrix}\beta_i\\ \alpha_i\end{smallmatrix}\right\}$, and every curve of $\left\{\begin{smallmatrix}\beta_i\\ \alpha_i\end{smallmatrix}\right\}$ except the $\left\{\begin{smallmatrix}\text{first}\\ \text{last}\end{smallmatrix}\right\}$ one, lies entirely within $\left\{\begin{smallmatrix}b_i\\ a_i\end{smallmatrix}\right\}$, (4) every curve of $\left\{\begin{smallmatrix}\beta_i\\ \alpha_i\end{smallmatrix}\right\}$ is within some circle of radius 1/2. There exists within p_i $(1 \leqq i \leqq m)$ an arc-curve chain ρ_i covering all those points of M that are within p_i and such that each curve of ρ_i is within some circle of radius 1/2. For every i $(2 \leqq i \leqq n)$ there exists on the first curve of α_i a point X_i not lying on the first arc of α_i. There exists an arc X_iD_i that lies, except for the point D_i, entirely within a_i and has no point except X_i in common with any arc or curve of α_i. For every i

* An arc-curve chain is a finite set K of closed curves a_1, a_2, $\cdots$ a_n and arcs $A_1'A_2$, $A_2'A_3$, $A_3'A_4$, $\cdots$ $A_{n-1}'A_n$, such that (1) every curve of K is without every other curve of K, (2) no two arcs of K have a point in common, (3) for every i $(1 \leqq i \leqq n - 1)$ the arc $A_i'A_{i+1}$ lies entirely without every curve of K except that A_i' is on a_i and A_{i+1} is on a_{i+1}. The arc-curve chain K is said to cover the point-set M if every point of M is either within a curve or on an arc of K.

$(1 \leqq i \leqq n)$ there exists, on the last curve of β_i, a point Y_i not lying on the last arc of β_i. There exists an arc Y_iC_i that lies, except for the point C_i, entirely within b_i and has no point except Y_i in common with any arc or curve of β_i. For every i $(1 \leqq i \leqq m)$ there exists, on the first curve of ρ_i, a point W_i that does not lie on the first arc of ρ_i. There exists an arc W_iE_i that lies entirely within p_i except for the point E_i and has no point, except W_i, in common with any arc or curve of ρ_i. For every i $(1 \leqq i \leqq m - 1)$ there exists, on the last curve of ρ_i, a point Z_i that does not lie on the last arc of ρ_i. There exists an arc Z_iF_i that lies entirely within p_i, except for the point F_i, and has no point in common with the arc W_iE_i or any arc of ρ_i and no point, except Z_i, in common with any curve of ρ_i. We now have a new chain K_2 whose curves are the curves of the chains $\alpha_1, \alpha_2, \cdots, \alpha_n, \beta_1, \beta_2, \cdots, \beta_n, \rho_1, \rho_2, \cdots, \rho_m$ and whose arcs are the arcs of these chains together with the arcs $X_2D_2, X_3D_3, \cdots, X_nD_n, Y_1C_1, Y_2C_2, \cdots, Y_nC_n, W_1E_1, W_2E_2, \cdots, W_mE_m, Z_1F_1, Z_2F_2, \cdots, Z_{m-1}F_{m-1}$. Every point of M is on an arc or within a curve of K_1 and also on an arc or within a curve of K_2. Every curve of K_1 is within a circle of radius 1. Every curve of K_2 is within a circle of radius 1/2. Every point of every curve of K_2 is within or on a curve of K_1 and every arc of K_2 is either within a curve or on an arc of K_1. There exists a chain K_3 each of whose curves is within a circle of radius 1/3 and which has a relation to K_3 similar to the above described relation of K_2 to K_1. This process may be continued. Thus there exists an infinite sequence of arc-curve chains $K_1, K_2, K_3, \cdots$ such that, for every n, (1) each point of M is on an arc or within a curve of K_n, (2) each curve of K_n is within some circle of radius $1/n$, (3) every point of each curve of K_{n+1} is within or on some curve of K_n and each arc of K_{n+1} is either within a curve or on an arc of K_n. Let $\overline{K}_n$ denote the set of all points $[X]$ such that X is on an arc or within a curve of K_n. Let t denote the set of all points that are common to $\overline{K}_1, \overline{K}_2, \overline{K}_3, \cdots$. It is clear that t contains every point of M. Let a_{n1} and b_{n1} respectively denote the first and last curves of the chain K_n. The interiors of the closed curves $a_{11}, a_{21}, a_{31}, \cdots$ have in common only one point A and those of the curves $b_{11}, b_{21}, b_{31}, \cdots$ have in common only one point B. That t is a simple continuous arc from A to B may be proved with the use of methods similar to those employed in the proof of Theorem 15 of Foundations.

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